Assessing Conformance To A Radius Tolerance

by Herb Voelcker and Ed Morse

Here's a bread-and-butter problem whose solution is not quite obvious: how do you use a CMM to determine whether a radius feature is 'in-spec'?

The American Y14.5M-1994 tolerancing standard defines a radius tolerance as shown in Figure 1: the radius feature must lie wholly within a zone defined by the radius limits. Figure 2

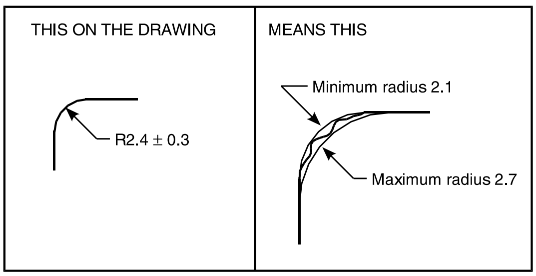

Figure 1: Specifying a radius

sets our problem[1] : we have CMM 'hits' on a toleranced radius feature, and we want to process the data to determine whether the feature is within radius limits (R_{Min}, R_{Max}).

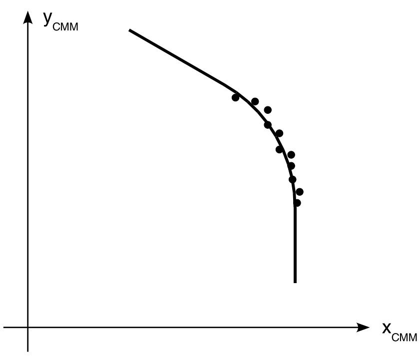

Figure 2: Measurement data

There are at least three approaches to this problem. To discuss these without a lot of mathematical clutter, we'll recast the problem from the CMM coordinate system of Figure 3 to the 'standard' coordinate system of Figure 4. In practical terms, this means that

- every point in CMM coordinates must be translated by ($-x_0$, $-y_0$), where (x_0, y_0) are the vertex coordinates shown in Fig. 3, and then rotated by $-(\phi + 90°)$. Alternatively,
- the data can be gathered in a local CMM coordinate system that is rotated by $(\phi + 90°)$ relative to the master CMM coordinate system, and has an origin at (x_0, y_0).

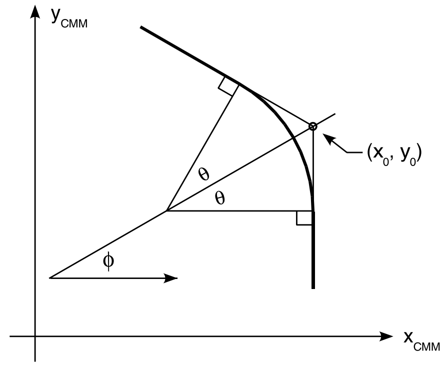

Figure 3: The radius feature in CMM coordinates

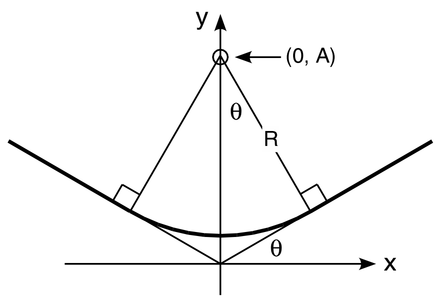

Figure 4: The radius feature in 'standard' coordinates

We shall see that the angle 2θ subtended by the radius feature is a critical parameter. Now to the problem, cast in the coordinates of Fig. 4.

Solution 1: Curve-fitting

The simplest solution is to fit a 2θ arc of unspecified radius to the data, using the least-squares fitting procedures supplied with most CMM software packages. The radius R_{Fit} of the arc can be regarded as the 'actual value' of the feature, and can be tested against the limits (R_{Min}, R_{Max}). Unfortunately this test

[1] There are several kinds of radius features – tangent cylindrical, cylindrical with located center, tangent spherical, and more. Fig. 1 and this article address only cylindrical tangency, and only in two dimensions for simplicity. A proper analysis should include 3-D issues.

does not conform to the zone-containment criterion specified in Fig. 1, and in fact can declare a dataset 'in-spec' when none of its points lie in the zone! (This is obvious if one regards the arc in Fig. 2 as a fitted arc that is fortuitously tangent to the adjacent faces… 'fortuitously' because only a few fitting routines support a tangency constraint.) A Minimax radius-fitting criterion (not usually available in standard CMM software packages) could meet the Fig. 1 specification, but then the curve-fitting approach would be a variant of Solution 3 (or an extended Solution 2) below.

Solution 2: Zone Containment

Direct implementation of the Fig. 1 criterion—that is, testing every point for containment in the zone—is the most literal approach to the problem. A containment test for a point p_i, with coordinates (x_i, y_i), can be constructed as follows.

- See Figure 5: the zone's bounding arcs lie on circles defined by

$$(A_{Max} - y)^2 + x^2 - R_{Max}^2 = 0 \,, \qquad (1a)$$

$$(A_{Min} - y)^2 + x^2 - R_{Min}^2 = 0 \,, \qquad (1b)$$

$$\text{where} \quad A_{Min|Max} = \frac{R_{Min|Max}}{\cos\theta} \,. \qquad (1c)$$

- Fig. 5 shows that the zone can be split into a central region where the boundaries are arcs, and left and right "tail" regions where the boundaries are an arc and a line.

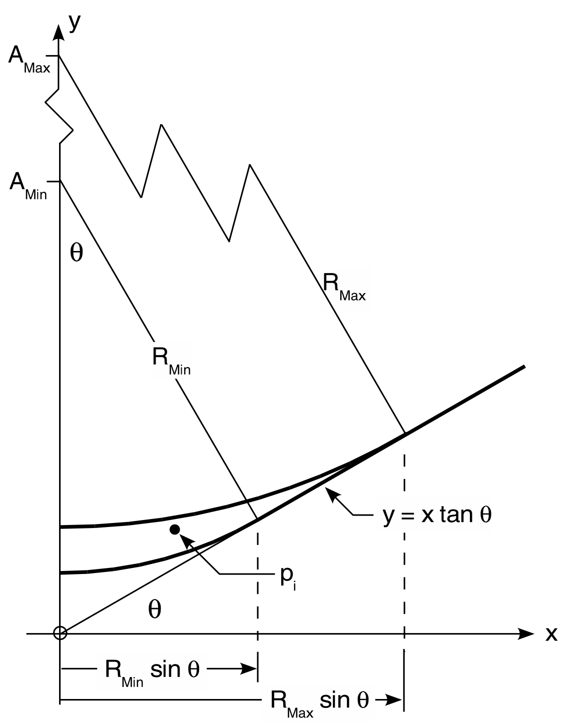

Figure 5: Half of the containment zone.

- Thus the containment test for a point p_i in the valid range $[-R_{Max}\sin\theta$ *to* $R_{Max}\sin\theta]$ is:

  when $0 \le |x_i| \le R_{Min}\sin\theta$ (the central region):

$$(A_{Min} - \sqrt{R_{Min}^2 - x_i^2}) \le y_i \le (A_{Max} - \sqrt{R_{Max}^2 - x_i^2}) \,; \qquad (2a)$$

  when $R_{Min}\sin\theta \le |x_i| \le R_{Max}\sin\theta$ (tail regions):

$$|x_i|\tan\theta \le y_i \le (A_{Max} - \sqrt{R_{Max}^2 - x_i^2}) \,. \qquad (2b)$$

Every in-range point in the dataset must pass this test if the radius feature is to conform to the tolerance. This approach clearly yields a valid conformance test because it implements the Y14.5 criterion directly, and a procedure similar to that in Solution 3 can be designed to apply the test to the dataset. The error sensitivity of the procedure can be studied through partial derivatives, as discussed briefly below.

Solution 3: Induced Radii

This solution associates a unique arc with each in-range point in the dataset, and then tests each arc's radius for inclusion in the zone. A unique arc can be associated with each point because we know the locus of the arcs' centers (the y-axis in Figures 4 and 5), and that the arcs must be tangent to the part's linear faces; thus only one point—a data point p_i—is needed to determine the arc, and hence the radius R_i associated with p_i. The relevant equations follow from Figures 4 or 5.

$$R_i = A_i \cos\theta \qquad (3a)$$

$$R_i^2 = x_i^2 + (A_i - y_i)^2 \qquad (3b)$$

These equations can be solved with the Quadratic Formula to obtain

$$R_i = \frac{\cos\theta}{\sin^2\theta}\left(y_i + \sqrt{y_i^2\cos^2\theta - x_i^2\sin^2\theta}\,\right) \qquad (4)$$

(The Quadratic Formula yields two roots, with (4) being the larger. The smaller root corresponds to an irrelevant smaller circle.)

The following procedure uses (4) to test the dataset for conformance and to report maximum and minimum 'actual values'.

```
Input R_Min, R_Max, and θ from the PartSpec;
BigR = 0; {initialize a working variable}
SmallR = BigNumber; {e.g. 10^6; another working variable}
For each data point do
  if (| x_i | ≤ R_Max sinθ ) then {valid range check}
    if (y_i < | x_i tanθ | ) then EXIT: Out-of-Tolerance. {point
    is below y = x tanθ line}
    Calculate R_i per Equation (4);
    if not (R_Min ≤ R_i ≤ R_Max) then EXIT: Out-of-Tolerance.
    if (R_i < SmallR) then SmallR ← R_i ; {search for smallest
    radius}
    if (R_i > BigR) then BigR ← R_i ; {search for largest radius}
  else Ignore_Point: Out-of-Range;
  End_do;
Report In-Tolerance and SmallR, BigR values.
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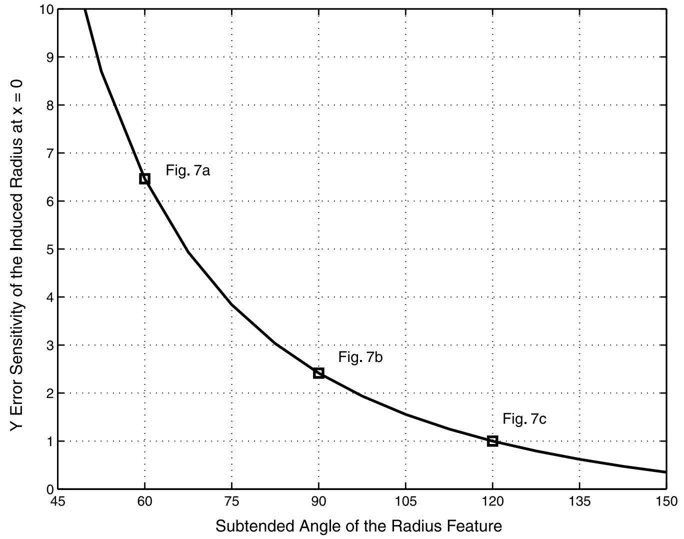

Figure 6: Measurement error sensitivity.

This procedure returns radius values that define the tightest zone that will bound the dataset, and its inclusion testing is equivalent to that in Solution 2—hence the procedure is valid.

Readers should be aware that (4) computes large numbers from small numbers (relatively large radii from relatively small point coordinates), and that data errors are multiplied commensurately. Data-error effects can be studied through the use of local expansions of the form

$$\Delta R_i = \sqrt{\left(\frac{\partial R_i}{\partial x_i}\Delta x_i\right)^2 + \left(\frac{\partial R_i}{\partial y_i}\Delta y_i\right)^2} \,, \qquad (5)$$

where the Δ terms are small changes from the exact values of R_i, x_i, y_i and the partial derivatives are evaluated at (x_i, y_i). The partial derivatives for (4) are

$$\frac{\partial R_i}{\partial x_i} = -\frac{x_i\cos\theta}{\sqrt{y_i^2\cos^2\theta - x_i^2\sin^2\theta}} \qquad (6a)$$

$$\frac{\partial R_i}{\partial y_i} = \frac{\cos\theta}{\sin^2\theta}\left(1 + \frac{y_i\cos^2\theta}{\sqrt{y_i^2\cos^2\theta - x_i^2\sin^2\theta}}\right) . \qquad (6b)$$

For example: at the 'nose' of a radius feature, where $x_i = 0$, only y-errors contribute to radius errors and (5) simplifies to

$$\Delta R_i = \left(\frac{\cos\theta\,(1 + \cos\theta)}{\sin^2\theta}\right)\Delta y_i \,. \qquad (7)$$

Figure 6 is a graph of ($\Delta R_i/\Delta y_i$), which can be termed the y-error sensitivity, versus the subtended angle of the radius feature. The labeled points on the graph correspond to the 60°, 90°, and 120° radius features shown in Figure 7. Note, for example, that for a data point near the nose of a 60° feature, an error in the y value results in an error 6.5X larger in the R_i value. (Although the graph does not show it, the multiplier grows as the data point moves toward the tails of the tolerance zone.) The error percentage does not grow, however, and the conformance-testing effectiveness of Solution 3 continues to be equivalent to that of Solution 2 when both process the same set of imprecise data.[2]

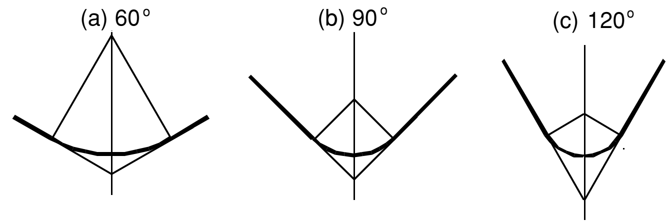

Figure 7: Radius features of varying 'sharpness'.

We have assumed throughout that the feature's subtended angle, 2θ, is known precisely for the part being checked. If the angle must be determined by measurement, an additional and complicated set of error sensitivities must be studied. Finally, we have said nothing about dataset sizes, which have a strong effect on the reliability and repeatability of conformance tests. This is a complicated matter that depends strongly on statistical assumptions (amongst other things).

We are grateful to Walter M. Stites, of AccraTronics Seals Corporation (and also the Y14.5 Sub-committee) for posing the radius problem to us as an important real problem, and for serving as both a thoughtful critic and a contributor to this manuscript.

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[2] All points in the conformance zone of Solution 2 map through (4) into equivalent points in the zone in "induced radius space" used in Solution 3, and hence perturbed points in one zone simply map into perturbed points in the other.