An Approximation of an Actual Value for a Position Specification

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This brief paper will discuss the development of an approximation scheme for the actual value of a GD&T position specification. First, a simple approximation will be proposed. Next, the approximation will be generalized and enhanced to determine an optimal approximation scheme.

1 Preliminaries

The language of GD&T defines several geometric tolerances. One of the governing standards, ASME Y14.5.1, proceeds to define actual values for each of the geometric tolerances. The geometric tolerance of interest here is that of position. The scope of this paper is limited to the 2-dimensional case. When used to control the resolved geometry of a feature, the actual value of position is twice the Cartesian distance from the theoretical resolved geometry to the measured resolved geometry at the worst-case. Symbolically, this can be expressed as

$$\oplus(\Delta x, \Delta y) = 2\sqrt{(\Delta x)^2 + (\Delta y)^2}$$

where Δx and Δy represent the deviation from nominal, or true, position in the x and y axes, respectively.

2 Simple Approximation

It has been found that $2\max(\Delta x, \Delta y) + \frac{1}{2}\min(\Delta x, \Delta y)$ is a fairly close approximation to $\oplus(\Delta x, \Delta y)$. Without loss of generality, assume that $\Delta x \geq 0$ and $\Delta y \geq 0$. To avoid the min/max notation, assume that $\Delta x \geq \Delta y$. Thus, there exists $k \in [0, 1]$ such that $\Delta y = k\Delta x$. Since $\Delta x \geq \Delta y$, $\max(\Delta x, \Delta y) = \Delta x$ and $\min(\Delta x, \Delta y) = \Delta y$, so the approximation becomes $2\Delta x + \frac{1}{2}\Delta y$. Since $\Delta y = k\Delta x$, the approximation becomes a function of k, namely

$$f(k) = 2\Delta x + \frac{1}{2}k\Delta x = \left(2 + \frac{1}{2}k\right)\Delta x$$

The error in the approximation can be modeled as a function of k, namely $E(k)$, and is calculated as the difference between the actual value and the approximate value.

$$\begin{aligned} E(k) &= \oplus(\Delta x, \Delta y) - f(k) \\ E(k) &= 2\sqrt{(\Delta x)^2 + (\Delta y)^2} - \left(2 + \frac{1}{2}k\right)\Delta x \\ E(k) &= 2\Delta x\sqrt{1 + k^2} - \left(2 + \frac{1}{2}k\right)\Delta x \\ E(k) &= \left(2\sqrt{1 + k^2} - 2 - \frac{1}{2}k\right)\Delta x \end{aligned}$$

Examination of the error function shows that there are two roots, namely $k = 0$ and $k = \frac{8}{15}$. Thus, there is no error in the approximation if $\Delta y = 0$ or if $\Delta y = \frac{8}{15}\Delta x$. $E(k)$ has one local minimum in the domain $[0, 1]$ at $k = \frac{1}{\sqrt{15}}$. The error at this point is

$$E\left(\frac{1}{\sqrt{15}}\right) = \left(\frac{\sqrt{15}}{2} - 2\right)\Delta x \approx -0.064\Delta x$$

The error at the local minimum is approximately 6.4% of Δx. Evaluating the error function at the boundaries of the domain yields

$$E(0) = 0$$
$$E(1) = \left(2\sqrt{2} - \frac{5}{2}\right)\Delta x \approx 0.328\Delta x$$

Thus, this approximation scheme has up to approximately 32.8% error when $k = 1$, or $\Delta y = \Delta x$. This can be demonstrated with an example. Assume that $\Delta x = 0.010 = \Delta y$. The actual position value is equal to

$$\oplus(0.010, 0.010) = 2\sqrt{(0.010)^2 + (0.010)^2} \approx 0.0283$$

Using the proposed approximation scheme with $k = 1$, the approximate position value is

$$f(1) = \left(2 + \frac{1}{2}\right)(0.010) = 0.025$$

The difference between the actual position value and the approximate position value is approximately $0.0283 - 0.025 = 0.0033$ which is 33% of $\Delta x = 0.010$.

A graph of the error function $E(k)$ can be seen below.

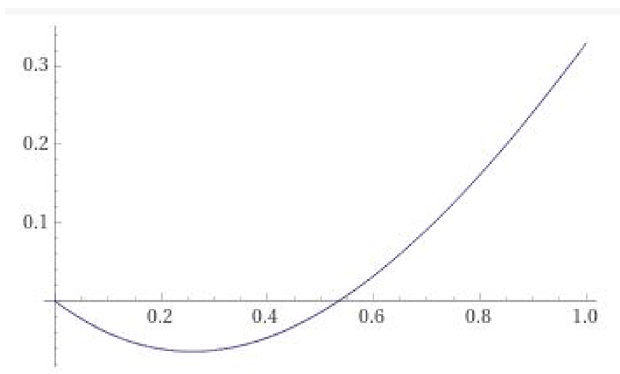

Figure 1: Error function $E(k)$ before optimization

3 Optimization of the Approximation Scheme

To optimize the approximation scheme, assume that the approximation is given by $\alpha \max(\Delta x, \Delta y) + \beta \min(\Delta x, \Delta y)$. Note that $\alpha = 2$ and $\beta = \frac{1}{2}$ will yield the approximation scheme given above. Assume that $\Delta x \geq \Delta y$ such that there exists $k \in [0, 1]$ with $\Delta y = k\Delta x$. Then, the approximation becomes a function of k.

$$f(k) = \alpha\Delta x + \beta k\Delta x = (\alpha + \beta k)\Delta x$$

To optimize the error function $E(k) = \oplus(k) - f(k) = 2\Delta x\sqrt{1 + k^2} - (\alpha + \beta k)\Delta x$, the following optimization conditions are imposed:

$$E(0) = -E(k^*) = E(1) \tag{1}$$

$$\left.\frac{dE(k)}{dk}\right|_{k=k^*} = 0 \tag{2}$$

for $k^* \in [0, 1]$. These two conditions ensure that the error is distributed more evenly about the lines $x = k^*$ and $y = 0$, which will ensure that the magnitude of the error is smaller overall. Working from the first condition, the error function can be evaluated at the boundaries of the interval $[0, 1]$.

$$\begin{aligned} E(0) &= 2\Delta x - \alpha \Delta x \\ E(1) &= 2\sqrt{2}\Delta x - \alpha \Delta x - \beta \Delta x \end{aligned}$$

and thus, since the error is conditionally equal at the boundaries,

$$\beta = 2\left(\sqrt{2} - 1\right) \approx 0.828$$

The second optimization condition is used to determine the minimum of the error function ($k = k^*$) through differentiation.

$$\begin{aligned} \left.\frac{dE(k)}{dk}\right|_{k=k^*} &= \left.\frac{d}{dk}\left(2\Delta x\sqrt{1+k^2} - (\alpha + \beta k)\Delta x\right)\right|_{k=k^*} \\ &= 2k^*\Delta x(1 + {k^*}^2)^{-1/2} - \beta \Delta x \end{aligned}$$

Since the minimum is desired, the derivative at $k = k^*$ must be equal to 0. This implies

$$\begin{aligned} \beta \Delta x &= 2k^*\Delta x(1 + {k^*}^2)^{-1/2} \\ \beta^2 &= \frac{4{k^*}^2}{1 + {k^*}^2} \end{aligned}$$

Solving for k^*, it can be shown that

$$k^* = \frac{1}{2}\sqrt{\beta} = \frac{1}{2}\sqrt{2\left(\sqrt{2} - 1\right)} \approx 0.455$$

Now that the local minimum is known to be at $k = \frac{1}{2}\sqrt{\beta}$, the first optimization condition can be used for a second time.

$$\begin{aligned} E(0) &= -E(k^*) \\ 2\Delta x - \alpha \Delta x &= -2\Delta x\sqrt{1 + \frac{1}{4}\beta} + \alpha \Delta x + \frac{1}{2}\beta^{3/2}\Delta x \\ 2 - \alpha &= -2\sqrt{1 + \frac{1}{4}\beta} + \alpha + \frac{1}{2}\beta^{3/2} \end{aligned}$$

Solving for α, it can be shown that

$$\begin{aligned} \alpha &= 1 + \sqrt{1 + \frac{1}{4}\beta} - \frac{1}{4}\beta^{3/2} \\ &= 1 + \sqrt{\frac{1+\sqrt{2}}{2}} - \frac{1}{4}\left(2\left(\sqrt{2} - 1\right)\right)^{3/2} \\ &\approx 1.910 \end{aligned}$$

With α and β determined, the approximation can be written as

$$\begin{aligned} f(k) &= (\alpha + \beta k)\Delta x \\ f(k) &= \left(1 + \sqrt{\frac{1+\sqrt{2}}{2}} - \frac{1}{4}\left(2\left(\sqrt{2} - 1\right)\right)^{3/2} + 2\left(\sqrt{2} - 1\right)k\right)\Delta x \end{aligned}$$

The error function $E(k)$ can be analyzed to determine the error allowed through this updated approximation scheme by evaluating the error function at only one of the limits of the domain, since the approximation was determined by setting $E(0) = -E(k^*) = E(1)$. For simplicity, $E(0)$ will be evaluated.

$$
\begin{aligned}
&E(k) = \oplus(k) - f(k) \\
&E(k) = 2\Delta x\sqrt{1+k^2} - \left(1 + \sqrt{\frac{1+\sqrt{2}}{2}} - \frac{1}{4}\left(2\left(\sqrt{2}-1\right)\right)^{3/2} + 2\left(\sqrt{2}-1\right)k\right)\Delta x \\
&E(0) = 2\Delta x - \left(1 + \sqrt{\frac{1+\sqrt{2}}{2}} - \frac{1}{4}\left(2\left(\sqrt{2}+1\right)\right)^{3/2}\right)\Delta x \\
&E(0) = \left(1 - \sqrt{\frac{1+\sqrt{2}}{2}} + \frac{1}{4}\left(2\left(\sqrt{2}-1\right)\right)^{3/2}\right)\Delta x \\
&E(0) \approx 0.090\Delta x
\end{aligned}
$$

Thus, the error is approximately 9.0% of Δx using the refined approximation scheme. The new scheme guarantees that the approximate value obtained will be within $0.09\Delta x$ of the actual position value. This is significantly better than the original proposal ($\approx 0.33\Delta x$ error). A graph of the error function $E(k)$ can be seen below.

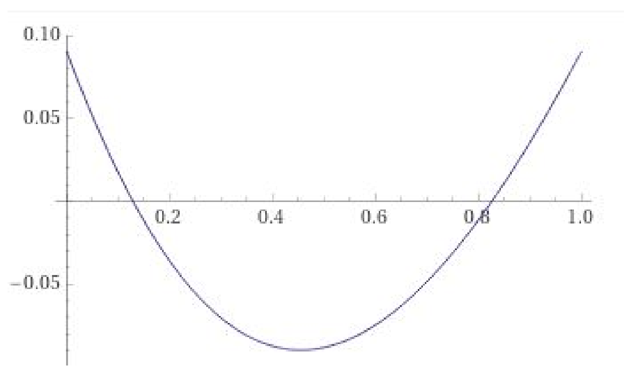

Figure 2: Error function $E(k)$ after optimization

The approximation $f(k)$ can be written in terms of Δx and Δy using the fact that $\Delta y = k\Delta x$. Let $\oplus_{approx}(\Delta x, \Delta y)$ be the approximate value of the position given deviations Δx and Δy. Then,

$$
\begin{aligned}
&\oplus_{approx}(\Delta x, \Delta y) = \left(1 + \sqrt{\frac{1+\sqrt{2}}{2}} - \frac{1}{4}\left(2\left(\sqrt{2}-1\right)\right)^{3/2}\right)\max(\Delta x, \Delta y) + 2\left(\sqrt{2}-1\right)\min(\Delta x, \Delta y) \\
&\oplus_{approx}(\Delta x, \Delta y) \approx 1.910\max(\Delta x, \Delta y) + 0.828\min(\Delta x, \Delta y)
\end{aligned}
$$

4 Conclusion

Two approximations of the actual position value have been proposed in this paper, one being simpler than the other but yielding more error. The simpler approximation yields error up to approximately 33% of $\max(\Delta x, \Delta y)$ and is of the form

$$\oplus_{approx}(\Delta x, \Delta y) = 2\max(\Delta x, \Delta y) + 0.5\min(\Delta x, \Delta y)$$

while the optimized approximation yields error up to approximately 9% of $\max(\Delta x, \Delta y)$ and is of the form

$$\bigoplus_{approx}(\Delta x, \Delta y) \approx 1.910 \max(\Delta x, \Delta y) + 0.828 \min(\Delta x, \Delta y)$$

Both approximations were obtained by assuming that the best approximation will be of the form

$$\bigoplus_{approx}(\Delta x, \Delta y) = \alpha \max(\Delta x, \Delta y) + \beta \min(\Delta x, \Delta y)$$

for some $\alpha, \beta \in \mathbb{R}$. There may be another form that better approximates the actual value and the exercise of discovering it is left to the reader.